Journal of Sound and Vibration (2001) **242**(1), 1–7
doi:10.1006/jsvi.2000.3365, available online at http://www.idealibrary.com on IDEAL®

LARGE-AMPLITUDE FREE VIBRATION OF A CONSERVATIVE SYSTEM WITH INERTIA AND STATIC NON-LINEARITY

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(*Received* 6 *August* 1999, *and in final form* 22 *May* 2000)

A power-series solution is presented for the free vibrations of a conservative oscillator having inertia and static non-linearities. The periodic vibrations of the oscillator are captured by the power-series method upon transforming the time variable into a harmonically oscillating time. A recursive relation is established between the solution coefficients which depend on initial conditions and oscillating time frequency. Rayleigh's energy principle is then used to determine the oscillating time frequency. The results show excellent agreement for the vibration frequency with numerical solutions even for relatively large-vibration amplitudes

1. INTRODUCTION

The free vibration analysis of engineering structures undergoing large-amplitude oscillations often involves discretizing the structure and results in a temporal problem having inertia and/or static non-linearities. In general, such problems are not amenable to exact treatment and approximate techniques must be resorted to. Amongst these, the perturbation method whose application has recently been extended to oscillators with strong non-linearity [1–3] is in common use. However, the algebraic manipulations inherent in the perturbation procedure involve excessive labour and this prompted the use of recently developed symbolic software to ease the computational burden.

Recently, a power-series method has been developed and applied to several free and forced vibration oscillators having cubic non-linearity of the static type [4–6]. In this paper, the use of the power-series method is extended to a conservative oscillator with inertia and static non-linearities [7], which simulates the uni-modal large-amplitude free vibration of a cantilever beam carrying an intermediate lumped mass with a rotary inertia. A significant advantage of the present method is the minimum algebraic manipulations and computer coding involved.

2. FORMULATION

Consider the non-linear oscillator

$$\ddot{u} + u + \alpha u^2 \ddot{u} + \alpha \dot{u}^2 u + \beta u^3 = 0 \tag{1}$$

subject to the initial conditions $u(0) = u_0$, $\dot{u}(0) = v_0$. The over-dot denotes differentiation with respect to time t. This system [7] describes the uni-modal large-amplitude free vibrations of a slender inextensible cantilever beam carrying a lumped mass and rotary inertia at an intermediate position along its span. The third and fourth terms in equation (1) represent inertia-type cubic non-linearity arising from the inextensibility assumption. The last term is a static-type cubic non-linearity associated with the potential energy stored in bending. The modal constants α and β result from the discretization procedure and have specific values for each mode.

A power-series analysis of the periodic motion of undamped non-linear oscillators, starts by transforming the time variable t into a harmonically oscillating time τ such that

$$\tau = \sin \omega t \tag{2}$$

which oscillates between the values -1 and $+1$ at a constant frequency ω, to be determined, as t is increased indefinitely. Introducing equation (2) into equation (1), the transformed problem becomes

$$\omega^2(1 + \alpha u^2)((1 - \tau^2)u'' - \tau u') + u + \alpha\omega^2(1 - \tau^2)u'^2 u + \beta u^3 = 0, \tag{3}$$

subject to the initial conditions $u(0) = u_0$, $u'(0) = v_0/\omega$, where the prime denotes differentiation with respect to τ. In the absence of non-linear terms, the linear theory of differential equations [8] assures a power-series expansion of equation (3) about $\tau = 0$ that converges for $|\tau| < 1$. This effectively covers the entire time domain except at the singular points $\tau = \pm 1$. Here, it will be assumed that a convergent power-series expansion for equation (3) exists in the form

$$u(\tau) = a_1 + a_2\tau + a_3\tau^2 + \cdots = \sum_{n=1}^{\infty} a_n \tau^{n-1}, \tag{4}$$

where a_i are constant coefficients to be determined. By using equation (4), the various terms in equation (3) can conveniently be written as

$$\begin{aligned} u^3 &= \sum_{n=1}^{\infty} c_n \tau^{n-1}, \\ (1 - \tau^2)u'^2 u &= \sum_{n=1}^{\infty} g_n \tau^{n-1}, \\ 1 + \alpha u^2 &= \sum_{n=1}^{\infty} f_n \tau^{n-1}, \\ (1 - \tau^2)u'' - \tau u' &= \sum_{n=1}^{\infty} d_n \tau^{n-1}, \end{aligned} \tag{5}$$

where $d_n = n(n + 1)a_{n+2} - (n - 1)^2 a_n$ and c_n, g_n, f_n, and d_n are constant coefficients that can be computed once the constants $a_1, a_2, \ldots, a_n$ are known. Introducing equation (5) into equation (3) gives

$$\sum_{n=1}^{\infty} \left[\omega^2 \left(f_1 d_n + \sum_{k=1}^{n-1} d_k f_{n-k+1} \right) + a_n + \alpha\omega^2 g_n + \beta c_n \right] \tau^{n-1} = 0. \tag{6}$$

Equation (6) is satisfied exactly for all values of time if the square bracketed coefficient of each power term is made to vanish. This introduces the recurrence relation between the solution coefficients

$$a_{n+2} = \frac{(\omega^2 f_1 (n-1)^2 - 1)a_n - \alpha\omega^2 g_n - \beta c_n - \omega^2 \sum_{k=1}^{n-1} d_k f_{n-k+1}}{f_1 n(n+1)\omega^2}, \quad n = 1, 2, \ldots . \tag{7}$$

The summation in equation (7) applies for $n > 1$ and, therefore, is set to zero when $n = 1$.

The evaluation of the solution coefficients begins by considering the initial conditions of the motion, which is conveniently assumed to start from the maximum displacement position with zero velocity. Applying these conditions to equation (4) gives

$$a_1 = u_0, \quad a_2 = 0{\cdot}0. \tag{8}$$

The starting values for various coefficients appearing in equation (7) can now be computed as

$$f_1 = 1 + \alpha u_0^2, \quad g_1 = 0{\cdot}0, \quad c_1 = u_0^3. \tag{9}$$

The remaining solution coefficients a_3, a_4 are computed recursively from equation (7) in conjunction with equation (5) for a specified value of the oscillating time frequency ω. It follows that the solution coefficients depend on the first two fundamental constants (a_1 and a_2) associated with initial conditions and on the oscillating time frequency which is an auxiliary parameter that remains to be determined. Because the system is conservative, this frequency can be obtained by enforcing Rayleigh's energy principle which stipulates equal maximum potential and kinetic energies. For the system under consideration, the maximum potential energy, which occurs at maximum displacement, is given by

$$V_{\max} = \tfrac{1}{2} u_0^2 + \tfrac{1}{4} \beta u_0^4. \tag{10}$$

The kinetic energy is written as

$$T = \tfrac{1}{2}(1 + \alpha u^2)\dot{u}^2 = \tfrac{1}{2}\omega^2(1 - \tau^2)(1 + \alpha u^2)u'^2 \tag{11}$$

and attains a maximum value at the equilibrium position for which $\omega t = \pi/4, 3\pi/4, 5\pi/4, \ldots$. From equation (2), this position is reached at $\tau = \pm 1/\sqrt{2}$.

Since τ is periodic, equation (4) is capable of capturing periodic motion. Moreover, a direct result of assuming the motion to start at $t = \tau = 0$ from maximum displacement position is the vanishing of all the coefficients of odd powers in equation (4). Subsequently, the same motion is captured every half-cycle (positive or negative) of the oscillating time. This requires the oscillating time frequency to be equal to one-half the vibration frequency (Ω), i.e.

$$\omega = \frac{\Omega}{2}. \tag{12}$$

3. RESULTS AND DISCUSSION

The amplitude-dependent vibration frequency of the non-linear oscillator, equation (1), was computed for a set of initial amplitudes by using the recurrence relation, equation (9) in conjunction with Rayleigh's energy principle. In order to validate the results, a numerical solution that employs the fourth order Runge–Kutta procedure was also obtained. In each case, the motion was assumed to start from the maximum displacement position with zero velocity. This condition determined the first two fundamental coefficients (a_1 and a_2) from

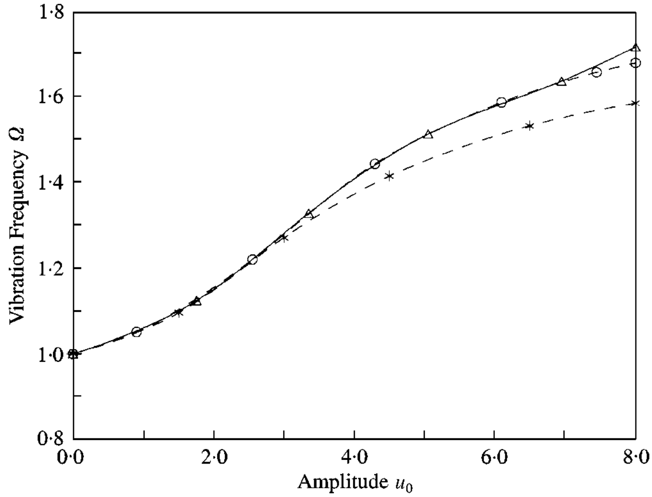

Figure 1. Vibration frequency versus amplitude u_0 ($v_0 = 0{\cdot}0$, $\alpha = 0{\cdot}1$, $\beta = 0{\cdot}2$) (△——△; present, ○----○; numerical, ∗-----∗; single HB).

equation (8). The remaining coefficients were computed recursively from equation (7) for an assumed value of the oscillating time frequency. A search for the actual frequency was conducted by computing the error function $\varepsilon = V_{max} - T_{max}$ for each ω and the actual frequency was obtained when $\varepsilon = 0$ which ensured that Rayleigh's energy principle was satisfied. For small amplitudes, this condition was reached when ε had a stationary minimum value whereas for large amplitudes, the error function changed sign at the correct frequency.

Figure 1 compares the vibration frequency variation with amplitude u_0 for $\alpha = 0{\cdot}1$ and $\beta = 0{\cdot}2$ with the results obtained using the numerical solution. Excellent agreement is seen between the two solutions except for a small discrepancy in the neighborhood of $u_0 = 8$. Also shown in the figure are the results of the harmonic balance method using single-mode approximation. The single-mode assumption $u(t) = u_0 \cos \Omega t$ results in a frequency–amplitude relation as follows:

$$\Omega^2 = \frac{1 + 0{\cdot}75\beta u_0^2}{1 + 0{\cdot}5\alpha u_0^2}. \tag{13}$$

Figure 2 demonstrates the convergence of the vibration frequency as the number of terms is increased. It is worth mentioning that for smaller amplitudes, the number of terms required to obtain an accurate solution is reduced. Figure 3 compares the computed vibration frequency with those obtained by the numerical method and harmonic balance method for various amplitudes with $\alpha = 1$ and $\beta = 1$. Again, similar agreement is seen between the power series and numerical solution.Due to the increased importance of the non-linear terms, the single-mode approximation becomes insufficient and, therefore, the two-mode approximation is used. In this case, with $u(0) = u_0$ and $\dot{u}(0) = 0$, the assumed solution takes the form

$$u(t) = A_1 \cos \Omega t + A_3 \cos 3\Omega t, \tag{14}$$

where $u_0 = A_1 + A_3$ is the total amplitude and A_1 and A_3 are the amplitudes of the fundamental and third harmonics, respectively. Upon substituting equation (14) into

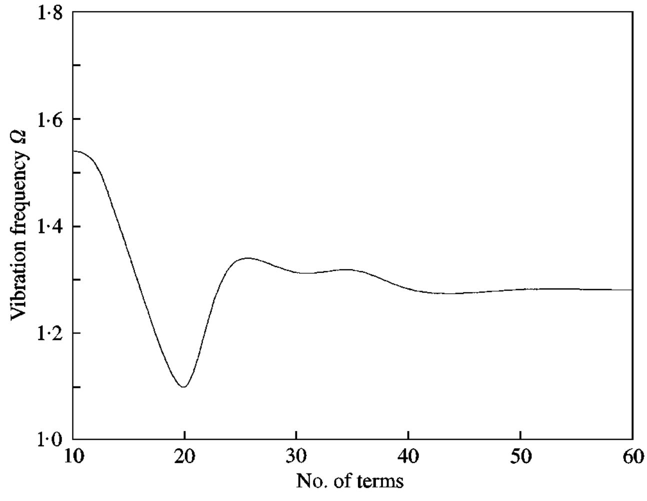

Figure 2. Convergence of vibration frequency ($u_0 = 3$, $\alpha = 0{\cdot}1$, $\beta = 0{\cdot}2$).

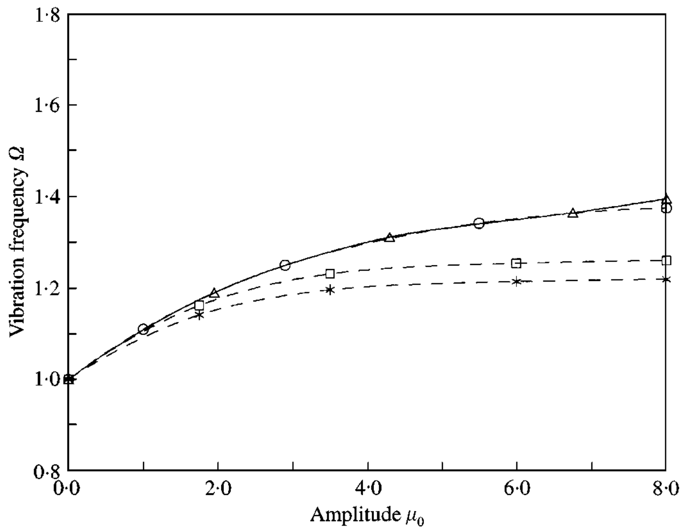

Figure 3. Vibration frequency versus amplitude u_0 ($v_0 = 0{\cdot}0$, $\alpha = 1$, $\beta = 1$) (△——△; present, ○----○; numerical, ∗------∗; single HB, □----□; two-mode HB).

equation (1), using trigonometric identities and equating the coefficients of the harmonics $\cos \Omega t$ and $\cos 3\Omega t$ to zero, the following two non-linear algebraic equations are obtained:

$$A_3 = \frac{0{\cdot}25\beta(A_1^3 + 3A_3^3) - 0{\cdot}5\alpha(A_1^3 + 9A_3^3)}{9\Omega^2 - 1 + 5\alpha\Omega^2 A_1^2 - 1{\cdot}5\beta A_1^2}, \tag{15}$$

$$\Omega^2 = \frac{1 + 0{\cdot}75\beta(A_1^2 + A_1A_3 + 2A_3^2)}{1 + 0{\cdot}5\alpha(A_1^2 + 3A_1A_3 + 10A_3^2)}. \tag{16}$$

TABLE 1

Odd-power series coefficients for two amplitudes

Amplitude		$i = 1$	$i = 3$	$i = 5$	$i = 7$
$u_0 = 2$	a_i	2·0000	− 3·8481	− 0·3621	− 0·0034
	a_{i+8}	0·1532	0·1574	0·0533	− 0·07306
	a_{i+16}	− 0·1328	− 0·0862	0·0299	0·1266
	a_{i+24}	0·1247	0·0172	− 0·1178	− 0·1697
$u_0 = 5$	a_i	5·0000	− 7·6600	− 2·9260	− 2·5350
	a_{i+8}	2·3780	− 1·9360	− 0·7489	1·8000
	a_{i+16}	0·0000	− 0·7489	1·8000	6·5110
	a_{i+24}	14·250	25·540	39·710	53·110

These two equations were solved iteratively. As shown in the Figure, the error in the single-mode harmonic balance is significant in this case because of the increased importance of the non-linear terms. The two-mode assumption is seen to give better results than the single one compared to those of the power-series method and the numerical method. It follows that the power-series solution can capture periodic motions of conservative systems which are far from the single-harmonic function of time.

Table 1 shows the odd-power-series coefficients for two amplitudes $u_0 = 2, 5$ with $\alpha = 0{\cdot}1$ and $\beta = 0{\cdot}2$. The number of terms used in each case was 45. For small amplitudes, a progressive decrease in the absolute value of the coefficients is obtained, whereas for large amplitudes, the solution coefficients increase in absolute value as shown in Table 1. This feature may be clarified by a ratio test. By noting that only even powers of τ exist in equation (4), the convergence of the solution is assured, providing the ratio between two consecutive terms $|a_{n+2}\tau^2/a_n| < 1$, so that $|a_{n+2}/a_n| < q$, where $q = 1/\tau^2$. For small-amplitude vibrations, convergent power-series solutions are obtained over the entire time domain corresponding to $|\tau| = 1$ for which $q = 1$ and the coefficients therefore decrease in absolute value with an increase of the index. For large-amplitude vibrations, convergent solutions are obtained [9] over one-quarter of the cycle corresponding to $|\tau| = 1/\sqrt{2}$ for which $q = 2$ and the series coefficients may therefore increase so that $|a_{n+2}/a_n| < 2$ as can be verified from Table 1 for $u_0 = 5$.

4. CONCLUSION

A power-series solution is presented for the large-amplitude free vibrations of an oscillator having inertia and static cubic non-linearities. The results show excellent agreement for the vibration frequency with a numerical solution even for relatively large amplitudes. A significant advantage of this method is that the computational effort involved is minimized and computer coding is simple. The method can be applied for free vibrations of conservative oscillators with quadratic or higher order inertia and static non-linearities.

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